In this chapter we will study functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$, functions which take vectors for inputs and give scalars for outputs. For example, the function that takes a point in space for input and gives back the temperature at that point is such a function; the function that reports the gross national product of a country is another such function. Note that the domain space of the first example is three-dimensional, while the domain of the latter has, for most countries, thousands of dimensions. As usual, whenever possible we will state our results for an arbitrary $n$-dimensional space, although most of our examples will deal with only two or three dimensions.

**Level sets and graphs**

We begin by considering some geometrical methods for picturing functions of the form $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

**Definition** Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a real number $c$, we call the set

$$L = \{(x_1, x_2, \ldots, x_n) : f(x_1, x_2, \ldots, x_n) = c\} \quad (3.1.1)$$

a *level set* of $f$ at level $c$. We also call $L$ a *contour* of $f$. When $n = 2$, we call $L$ a *level curve* of $f$ and when $n = 3$ we call $L$ a *level surface* of $f$. A plot displaying level sets for several different levels is called a *contour plot*.

**Example** Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$f(x, y) = 2x^2 + y^2.$$ 

Given a real number $c$, the set of all points satisfying

$$2x^2 + y^2 = c$$

is a level set of $f$. For $c < 0$, this set is empty; for $c = 0$, it consists of only the point $(0, 0)$; for any $c > 0$, the level set is an ellipse with center at $(0, 0)$. Hence a contour plot of $f$, as shown in Figure 3.1.1, consists of concentric ellipses.

**Example** Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$f(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}.$$
For any point \((x, y)\) on the circle of radius \(r > 0\) centered at the origin, \(f(x, y)\) has the constant value 
\[
\frac{\sin(r)}{r}.
\]

Hence a contour plot of \(f\), like that shown in Figure 3.1.2, consists of concentric circles centered at the origin.

**Example** Suppose \(f : \mathbb{R}^3 \to \mathbb{R}\) is defined by
\[
 f(x, y, z) = x^2 + 2y^2 + 3z^2.
\]

The level surface of \(f\) with equation
\[
x^2 + 2y^2 + 3z^2 = 1
\]
is shown in Figure 3.1.3. Note that, for example, fixing a value \(z_0\) of \(z\) yields the equation
\[
x^2 + y^2 = 1 - 3z_0^2,
\]
the equation of an ellipse. This explains why a slice of the level surface shown in Figure 3.1.3 parallel to the $xy$-plane is an ellipse. Similarly, slices parallel to the $xz$-plane and the $yz$-plane are ellipses, which is why this surface is an example of an ellipsoid.

Figure 3.1.3 The level surface $x^2 + 2y^2 + 3z^2 = 1$
Definition  Given a function \( f : \mathbb{R}^n \to \mathbb{R} \), we call the set
\[
G = \{ (x_1, x_2, \ldots, x_n, x_{n+1}) : x_{n+1} = f(x_1, x_2, \ldots, x_n) \}
\] (3.1.2)
the graph of \( f \).

Note that the graph \( G \) of a function \( f : \mathbb{R}^n \to \mathbb{R} \) is in \( \mathbb{R}^{n+1} \). As a consequence, we can picture \( G \) only if \( n = 1 \), in which case \( G \) is a curve as studied in single-variable calculus, or \( n = 2 \), in which case \( G \) is a surface in \( \mathbb{R}^3 \).
**Example**  Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 2x^2 + y^2.$$ 

The graph of $f$ is then the set of all points $(x, y, z)$ in $\mathbb{R}^3$ which satisfy the equation $z = 2x^2 + y^2$. One way to picture the graph of $f$ is to imagine raising the level curves in Figure 3.1.1 to their respective heights above the $xy$-plane, creating the surface in $\mathbb{R}^3$ shown in Figure 3.1.4. Another way to picture the graph is to consider slices of the graph lying above a grid of lines parallel to the axes in the $xy$-plane. For example, for a fixed value of $x$, say $x_0$, the set of points satisfying the equation $z = 2x_0^2 + y^2$ is a parabola lying above the line $x = x_0$. Similarly, fixing a value $y_0$ of $y$ yields the parabola $z = 2x^2 + y_0$ lying above the line $y = y_0$. If we draw these parabolas for numerous lines of the form $x = x_0$ and $y = y_0$, we obtain a wire-frame of the graph. The graph shown in Figure 3.1.4 was obtained by filling in the surface patches of a wire-frame mesh, the outline of which is visible on the surface. This surface is an example of a *paraboloid*.

![Graph of $f(x, y) = \sin(\sqrt{x^2 + y^2})/\sqrt{x^2 + y^2}$](image)

Figure 3.1.5 Graph of $f(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}$

**Example**  Although the graphs of many functions may be sketched reasonably well by hand using the ideas of the previous example, for most functions a good picture of its graph requires either computer graphics or considerable artistic skill. For example, consider the graph of

$$f(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}.$$ 

Using the contour plot, we can imagine how the graph of $f$ oscillates as we move away from the origin, the level circles of the contour plot rising and falling with the oscillations.
of
\[
\frac{\sin(r)}{r},
\]
where \( r = \sqrt{x^2 + y^2} \). Equivalently, the slice of the graph above any line through the origin will be the graph of

\[
z = \frac{\sin(r)}{r}.
\]

This should give you a good idea what the graph of \( f \) looks like, but, nevertheless, most of us could not produce the picture of Figure 3.1.5 without the aid of a computer. Notice that although \( f \) is not defined at \((0, 0)\), it appears that \( f(x, y) \) approaches 1 as \((x, y)\) approaches 0. This is in fact true, a consequence of the fact that

\[
\lim_{r \to 0} \frac{\sin(r)}{r} = 1.
\]

We will return to this example after we have introduced limits and continuity.

**Limits and continuity**

By now the following two definitions should look familiar.

**Definition** Let \( a \) be a point in \( \mathbb{R}^n \) and let \( O \) be the set of all points in the open ball of radius \( r > 0 \) centered at \( c \) except \( c \) itself. That is,

\[
O = \{ x : x \text{ is in } B^n(c, r), x \neq c \}.
\]

Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is defined for all \( x \) in \( O \). We say the limit of \( f(x) \) as \( x \) approaches \( c \) is \( L \), written

\[
\lim_{x \to c} f(x) = L,
\]

whenever \( \lim_{m \to \infty} x_m = c \).

**Definition** Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) is defined for all \( x \) in some open ball \( B^n(c, r) \), \( r > 0 \). We say \( f \) is continuous at \( c \) if

\[
\lim_{x \to c} f(x) = f(c).
\]

The following basic properties of limits follow immediately from the analogous properties for limits of sequences.

**Proposition** Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) with

\[
\lim_{x \to c} f(x) = L
\]

and

\[
\lim_{x \to c} g(x) = M.
\]


Then
\[
\lim_{x \to c} (f(x) + g(x)) = L + M, 
\]
(3.1.6)
\[
\lim_{x \to c} (f(x) - g(x)) = L - M, 
\]
(3.1.7)
\[
\lim_{x \to c} f(x)g(x) = LM, 
\]
(3.1.8)
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, 
\]
(3.1.9)
and
\[
\lim_{x \to c} kf(x) = kL 
\]
(3.1.10)
for any scalar \(k\).

Now suppose \(f : \mathbb{R}^n \to \mathbb{R}, \ h : \mathbb{R} \to \mathbb{R}, \)
\[
\lim_{x \to c} f(x) = L, 
\]
(3.1.11)
and \(h\) is continuous at \(L\). Then for any sequence \(\{x_m\}\) in \(\mathbb{R}^n\) with
\[
\lim_{m \to \infty} x_m = c, 
\]
(3.1.12)
we have
\[
\lim_{m \to \infty} f(x_m) = L, 
\]
(3.1.13)
and so
\[
\lim_{m \to \infty} h(f(x_m)) = h(L) 
\]
(3.1.14)
by the continuity of \(h\) at \(L\). Thus we have the following result about compositions of functions.

**Proposition** If \(f : \mathbb{R}^n \to \mathbb{R}, \ h : \mathbb{R} \to \mathbb{R}, \)
\[
\lim_{x \to c} f(x) = L, 
\]
and \(h\) is continuous at \(L\), then
\[
\lim_{x \to c} h \circ f(x) = \lim_{x \to c} h(f(x)) = h(L). 
\]
(3.1.15)

**Example** Suppose we define \(f : \mathbb{R}^n \to \mathbb{R}\) by
\[
f(x_1, x_2, \ldots, x_n) = x_k, 
\]
where \(k\) is a fixed integer between 1 and \(n\). If \(a = (a_1, a_2, \ldots, a_n)\) is a point in \(\mathbb{R}^n\) and \(\lim_{m \to \infty} x_m = a\), then
\[
\lim_{m \to \infty} f(x_m) = \lim_{m \to \infty} x_{mk} = a_k, 
\]
where $x_{mk}$ is the $k$th coordinate of $x_m$. Thus

$$\lim_{x \to a} f(x) = a_k.$$  

This result is a basic building block for the examples that follow. For a particular example, if $f(x, y) = x$, then

$$\lim_{(x, y) \to (2, 3)} f(x, y) = \lim_{(x, y) \to (2, 3)} x = 2.$$  

**Example** If we define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = xyz,$$

then, using (3.1.8) in combination with the previous example,

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = \lim_{(x, y, z) \to (a, b, c)} xyz = (\lim_{(x, y, z) \to (a, b, c)} x)(\lim_{(x, y, z) \to (a, b, c)} y)(\lim_{(x, y, z) \to (a, b, c)} z) = abc.$$  

for any point $(a, b, c)$ in $\mathbb{R}^3$. For example,

$$\lim_{(x, y, z) \to (3, 2, 1)} f(x, y, z) = \lim_{(x, y, z) \to (3, 2, 1)} xyz = (3)(2)(1) = 6.$$  

**Example** Combining the previous examples with (3.1.6), (3.1.7), (3.1.8), and (3.1.10), we have

$$\lim_{(x, y, z) \to (2, 1, 3)} (xy^2 + 3xyz - 6xz) = (\lim_{(x, y, z) \to (2, 1, 3)} x)(\lim_{(x, y, z) \to (2, 1, 3)} y)(\lim_{(x, y, z) \to (2, 1, 3)} z) + 3(\lim_{(x, y, z) \to (2, 1, 3)} x)(\lim_{(x, y, z) \to (2, 1, 3)} y)(\lim_{(x, y, z) \to (2, 1, 3)} z) - 6(\lim_{(x, y, z) \to (2, 1, 3)} x)(\lim_{(x, y, z) \to (2, 1, 3)} z) = (2)(1)(1) + (3)(2)(1)(3) - (6)(2)(3) = -16.$$  

The last three examples are all examples of polynomials in several variables. In general, a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x_1, x_2, \ldots, x_n) = a x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n},$$

where $a$ is a scalar and $i_1, i_2, \ldots, i_n$ are nonnegative integers, is called a *monomial*. A function which is a sum of monomials is called a *polynomial*. The following proposition is a consequence of the previous examples and (3.1.6), (3.1.7), (3.1.8), and (3.1.10).
Proposition  If \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is a polynomial, then for any point \( c \) in \( \mathbb{R}^n \),
\[
\lim_{x \to c} f(x) = f(c). \tag{3.1.16}
\]
In other words, \( f \) is continuous at every point \( c \) in \( \mathbb{R}^n \).

If \( g \) and \( h \) are both polynomials, then we call the function
\[
f(x) = \frac{g(x)}{h(x)} \tag{3.1.17}
\]
a rational function. The next proposition is a consequence of the previous theorem and (3.1.9).

Proposition  If \( f \) is a rational function defined at \( c \), then
\[
\lim_{x \to c} f(x) = f(c). \tag{3.1.18}
\]
In other words, \( f \) is continuous at every point \( c \) in its domain.

Example  Since
\[
f(x, y, z) = \frac{x^2y + 3xyz^2}{4x^2 + 3z^2}
\]
is a rational function, we have, for example,
\[
\lim_{(x, y, z) \to (2, 1, 3)} f(x, y, z) = \lim_{(x, y, z) \to (2, 1, 3)} \frac{x^2y + 3xyz^2}{4x^2 + 3z^2} = \frac{4 + 54}{16 + 27} = \frac{58}{43}.
\]

Example  Combining (3.1.18) with (3.1.15), we have
\[
\lim_{(x, y, z) \to (1, 2, 1)} \log \left( \frac{1}{x^2 + y^2 + z^2} \right) = \log \left( \lim_{(x, y, z) \to (1, 2, 1)} \frac{1}{x^2 + y^2 + z^2} \right) = \log \left( \frac{1}{6} \right) = -\log(6).
\]

From the continuity of the square root function and our result above about the continuity of polynomials, we may conclude that the function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) defined by
\[
f(x_1, x_2, \ldots, x_n) = \|(x_1, x_2, \ldots, x_n)\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}
\]
is a continuous function. This fact is useful in computing some limits, particularly in combination with the fact that for any point \( x = (x_1, x_2, \ldots, x_n) \) in \( \mathbb{R}^n \),
\[
\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \geq \sqrt{x_k^2} = |x_k| \tag{3.1.19}
\]
for any \( k = 1, 2, \ldots, n \).
Example  Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \frac{x^2 y}{x^2 + y^2}.$$

Although $f$ is a rational function, we cannot use (3.1.18) to compute

$$\lim_{(x,y) \to (0,0)} f(x,y)$$

since $f$ is not defined at $(0,0)$. However, if we let $x = (x,y)$, then, using (3.1.19),

$$|f(x,y)| = \left| \frac{x^2 y}{x^2 + y^2} \right| = \frac{|x|^2 |y|}{|x|^2 + |y|^2} = |x|^2 |y| \leq \frac{\|x\|^2 \|x\|}{\|x\|^2} = \|x\|.$$

Now

$$\lim_{(x,y) \to (0,0)} \|x\| = 0,$$

so

$$\lim_{(x,y) \to (0,0)} |f(x,y)| = 0.$$

Hence

$$\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2} = 0.$$

See Figure 3.1.6.
Recall that for a function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\lim_{t \to c} \varphi(t) = L$$

if and only if both

$$\lim_{t \to c^-} \varphi(t) = L$$

and

$$\lim_{t \to c^+} \varphi(t) = L.$$ 

In particular, if the one-sided limits do not agree, we may conclude that the limit does not exist. Similar reasoning may be applied to a function $f : \mathbb{R}^n \to \mathbb{R}$, the difference being that there are infinitely many different curves along which the variable $x$ might approach a given point $c$ in $\mathbb{R}^n$, as opposed to only the two directions of approach in $\mathbb{R}$. As a consequence, it is not possible to establish the existence of a limit with this type of argument. Nevertheless, finding two ways to approach $c$ which yield different limiting values is sufficient to show that the limit does not exist.

**Example** Suppose $g : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$g(x, y) = \frac{xy}{x^2 + y^2}.$$ 

If we define $\alpha : \mathbb{R}^2 \to \mathbb{R}$ by $\alpha(t) = (t, 0)$, then

$$\lim_{t \to 0} \alpha(t) = \lim_{t \to 0} (t, 0) = (0, 0)$$

and

$$\lim_{t \to 0} g(\alpha(t)) = \lim_{t \to 0} f(t, 0) = \lim_{t \to 0} \frac{0}{t^2} = 0.$$ 

Now $\alpha$ is a parametrization of the $x$-axis, so the previous limit computation says that $g(x, y)$ approaches 0 as $(x, y)$ approaches $(0, 0)$ along the $x$-axis. However, if we define $\beta : \mathbb{R} \to \mathbb{R}^2$ by $\beta(t) = (t, t)$, then $\beta$ parametrizes the line $x = y$,

$$\lim_{t \to 0} \beta(t) = \lim_{t \to 0} (t, t) = (0, 0),$$

and

$$\lim_{t \to 0} g(\beta(t)) = \lim_{t \to 0} f(t, t) = \lim_{t \to 0} \frac{t^2}{2t^2} = \frac{1}{2}.$$ 

Hence $g(x, y)$ approaches $\frac{1}{2}$ as $(x, y)$ approaches $(0, 0)$ along the line $x = y$. Since these two limits are different, we may conclude that $g(x, y)$ does not have a limit as $(x, y)$ approaches $(0, 0)$. Note that $g$ in this example and $f$ in the previous example are very similar functions, although our limit calculations show that their behavior around $(0, 0)$ differs significantly. In particular, $f$ has a limit as $(x, y)$ approaches $(0, 0)$, whereas $g$ does not. This may be
seen by comparing the graph of \( g \) in Figure 3.1.7, which has a tear at the origin, with that of \( f \) in Figure 3.1.6.

The next proposition lists some basic properties of continuous functions, all of which follow immediately from the similar list of properties of limits.

**Proposition** Suppose \( f : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) are both continuous at \( c \). Then the functions with values at \( x \) given by

\[
\begin{align*}
  f(x) + g(x), & \quad (3.1.20) \\
  f(x) - g(x), & \quad (3.1.21) \\
  f(x)g(x), & \quad (3.1.22) \\
  f(x) \quad & g(x) \\
  \quad \end{align*}
\]

(provided \( g(c) \neq 0 \)), and

\[
  kf(x), \quad (3.1.24)
\]

where \( k \) is any scalar, are all continuous at \( c \).

From the result above about the limit of a composition of two functions, we have the following proposition.

**Proposition** If \( f : \mathbb{R}^n \to \mathbb{R} \) is continuous at \( c \) and \( \varphi : \mathbb{R} \to \mathbb{R} \) is continuous at \( f(c) \), then \( \varphi \circ f \) is continuous at \( c \).
Example Since the function $\varphi(t) = \sin(t)$ is continuous for all $t$ and the function
\[
f(x, y, z) = \sqrt{x^2 + y^2 + z^2}
\]
is continuous at all points $(x, y, z)$ in $\mathbb{R}^3$, the function
\[
g(x, y, z) = \sin(\sqrt{x^2 + y^2 + z^2})
\]
is continuous at all points $(x, y, z)$ in $\mathbb{R}^3$.

Example Since the function
\[
h(x, y) = \sin(\sqrt{x^2 + y^2})
\]
is continuous for all $(x, y)$ in $\mathbb{R}^2$ (same argument as in the previous example) and the function
\[
g(x, y) = \sqrt{x^2 + y^2}
\]
is continuous for all $(x, y)$ in $\mathbb{R}^2$, the function
\[
f(x, y) = \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}
\]
is, using (3.1.23), continuous at every point $(x, y) \neq (0, 0)$ in $\mathbb{R}^2$. Moreover, if we let $x = (x, y)$, then
\[
\lim_{(x,y) \to (0,0)} f(x,y) = \lim_{(x,y) \to (0,0)} \frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{\sin(\|x\|)}{\|x\|} = \lim_{r \to 0} \frac{\sin(r)}{r} = 1.
\]
Hence the discontinuity at $(0,0)$ is removable. That is, if we define
\[
g(x,y) = \begin{cases} 
\frac{\sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0, 0), \\
1, & \text{if } (x, y) = (0, 0),
\end{cases}
\]
then $g$ is continuous for all $(x, y)$ in $\mathbb{R}^2$.

Open and closed sets
In single-variable calculus we talk about a function being continuous not just at a point, but on an open interval, meaning that the function is continuous at every point in the open interval. Similarly, we need to generalize the definition of continuity of a function $f : \mathbb{R}^n \to \mathbb{R}$ from that of continuity at a point in $\mathbb{R}^n$ to the idea of a function being continuous on a set in $\mathbb{R}^n$. Now the condition for a function $f$ to be continuous at a point $c$ requires that $f$ be defined on some open ball containing $c$. Hence, in order to say that $f$ is continuous at every point in some set $U$, it is necessary that, given any point $u$ in $U$, $f$
be defined on some open ball containing \( u \). This provides the motivation for the following definition.

**Definition** We say a set of points \( U \) in \( \mathbb{R}^n \) is *open* if whenever \( u \) is a point in \( U \), there exists a real number \( r > 0 \) such that the open ball \( B^n(u, r) \) lies entirely within \( U \). We say a set of points \( C \) in \( \mathbb{R}^n \) is *closed* if the set of all points in \( \mathbb{R}^n \) which do not lie in \( C \) form an open set.

**Example** \( \mathbb{R}^n \) is itself an open set.

**Example** Any open ball in \( \mathbb{R}^n \) is an open set. In particular, any open interval in \( \mathbb{R} \) is an open set. To see why, consider an open ball \( B^n(a, r) \) in \( \mathbb{R}^n \). Given a point \( y \) in \( B^n(a, r) \), let \( s \) be the smaller of \( \|y - a\| \) (the distance from \( y \) to the center of the ball) and \( r - \|y - a\| \) (the distance from \( y \) to the edge of the ball). Then \( B^n(y, s) \) is an open ball which lies entirely within \( B^n(a, r) \). Hence \( B^n(a, r) \) is an open set.

**Example** Any closed ball in \( \mathbb{R}^n \) is a closed set. In particular, any closed interval in \( \mathbb{R} \) is a closed set. To see why, consider a closed ball \( \bar{B}^n(a, r) \). Given a point \( y \) not in \( \bar{B}^n(a, r) \), let \( s = \|y - a\| - r \), the distance from \( y \) to the edge of \( \bar{B}^n(a, r) \). Then \( B^n(y, s) \) is an open ball which lies entirely outside of \( \bar{B}^n(a, r) \). Hence \( \bar{B}^n(a, r) \) is a closed set.

**Example** Given real numbers \( a_1 < b_1, a_2 < b_2, \ldots, a_n < b_n \), we call the set

\[
U = \{(x_1, x_2, \ldots, x_n) : a_i < x_i < b_i, i = 1, 2, \ldots, n\}
\]

an *open rectangle* in \( \mathbb{R}^n \) and the set

\[
C = \{(x_1, x_2, \ldots, x_n) : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}
\]

a *closed rectangle* in \( \mathbb{R}^n \). An argument similar to that in the previous example shows that \( U \) is an open set and \( C \) is a closed set.

**Definition** We say a function \( f : \mathbb{R}^n \to \mathbb{R} \) is *continuous on an open set* \( U \) if \( f \) is continuous at every point \( u \) in \( U \).

**Example** The function

\[
f(x, y, z) = \frac{3xyz - 6x}{x^2 + y^2 + z^2 + 1}
\]

is continuous on \( \mathbb{R}^3 \).

**Example** The functions

\[
f(x, y) = \begin{cases} 
\frac{x^2y}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0), \\
0, & \text{if } (x, y) = (0, 0),
\end{cases}
\]

and

\[
g(x, y) = \begin{cases} 
\sin(\sqrt{x^2 + y^2}) \left/ \sqrt{x^2 + y^2} \right., & \text{if } (x, y) \neq (0, 0), \\
1, & \text{if } (x, y) = (0, 0),
\end{cases}
\]

are, from our work in previous examples, continuous on \( \mathbb{R}^2 \).
Example The function
\[ g(x, y) = \frac{xy}{x^2 + y^2} \]
is continuous on the open set
\[ U = \{(x, y) : (x, y) \neq (0, 0)\}. \]

Note that in this case it is not possible to define \( g \) at \((0, 0)\) in such a way that the resulting function is continuous at \((0, 0)\), a consequence of our work above showing that \( g \) does not have a limit as \((x, y)\) approaches \((0, 0)\).

Example The function
\[ f(x, y) = \log(xy) \]
is continuous on the open set
\[ U = \{(x, y) : x > 0 \text{ and } y > 0\}. \]

Problems

1. Plot the graph and a contour plot for each of the following functions. Do your plots over regions large enough to illustrate the behavior of the function.
   (a) \( f(x, y) = x^2 + 4y^2 \)  
   (b) \( f(x, y) = x^2 - y^2 \)  
   (c) \( f(x, y) = 4y^2 - 2x^2 \)  
   (d) \( h(x, y) = \sin(x) \cos(y) \)  
   (e) \( f(x, y) = \sin(x + y) \)  
   (f) \( g(x, y) = \sin(x^2 + y^2) \)  
   (g) \( g(x, y) = \sin(x^2 - y^2) \)  
   (h) \( h(x, y) = xe^{-\sqrt{x^2+y^2}} \)  
   (i) \( f(x, y) = \frac{1}{2\pi} e^{-\frac{1}{\pi}(x^2+y^2)} \)  
   (j) \( f(x, y) = \sin(\pi \sin(x) + y) \)  
   (k) \( h(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2} \)  
   (l) \( g(x, y) = \log(\sqrt{x^2 + y^2}) \)

2. For each of the following, plot the contour surface \( f(x, y, z) = c \) for the specified value of \( c \).
   (a) \( f(x, y, z) = x^2 + y^2 + z^2, c = 4 \)  
   (b) \( f(x, y, z) = x^2 + 4y^2 + 2z^2, c = 7 \)  
   (c) \( f(x, y, z) = x^2 + y^2 - z^2, c = 1 \)  
   (d) \( f(x, y, z) = x^2 - y^2 + z^2, c = 1 \)

3. Evaluate the following limits.
   (a) \( \lim_{(x,y)\to(2,1)} (3xy + x^2y + 4y) \)  
   (b) \( \lim_{(x,y,z)\to(1,2,1)} \frac{3xyz}{2xy^2 + 4z} \)  
   (c) \( \lim_{(x,y)\to(2,0)} \frac{\cos(3xy)}{\sqrt{x^2+1}} \)  
   (d) \( \lim_{(x,y,z)\to(2,1,3)} ye^{2x-3y+z} \)
4. For each of the following, either find the specified limit or explain why the limit does not exist.

(a) \( \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^2} \)

(b) \( \lim_{(x,y) \to (0,0)} \frac{x}{x + y} \)

(c) \( \lim_{(x,y) \to (0,0)} \frac{x}{x + y^2} \)

(d) \( \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \)

(e) \( \lim_{(x,y) \to (0,0)} \frac{1 - e^{-(x^2+y^2)}}{x^2 + y^2} \)

(f) \( \lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^2 + y^2} \)

5. Let \( f(x, y) = \frac{x^2y}{x^4 + 4y^2} \).

(a) Define \( \alpha : \mathbb{R} \to \mathbb{R}^2 \) by \( \alpha(t) = (t, 0) \). Show that \( \lim_{t \to 0} f(\alpha(t)) = 0 \).

(b) Define \( \beta : \mathbb{R} \to \mathbb{R}^2 \) by \( \beta(t) = (0, t) \). Show that \( \lim_{t \to 0} f(\beta(t)) = 0 \).

(c) Show that for any real number \( m \), if we define \( \gamma : \mathbb{R} \to \mathbb{R}^2 \) by \( \gamma(t) = (t, mt) \), then \( \lim_{t \to 0} f(\gamma(t)) = 0 \).

(d) Define \( \delta : \mathbb{R} \to \mathbb{R}^2 \) by \( \delta(t) = (t, t^2) \). Show that \( \lim_{t \to 0} f(\delta(t)) = \frac{1}{5} \).

(e) What can you conclude about \( \lim_{(x,y) \to (0,0)} \frac{x^2y}{x^4 + 4y^2} \)?

(f) Plot the graph of \( f \) and explain your results in terms of the graph.

6. Discuss the continuity of the function

\[
f(x, y) = \begin{cases} 
1 - e^{-\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq (0, 0), \\
\frac{\sqrt{x^2 + y^2}}{x^2 + y^2} & \text{if } (x, y) = (0, 0).
\end{cases}
\]

7. Discuss the continuity of the function

\[
g(x, y) = \begin{cases} 
\frac{x^2y^2}{x^4 + y^4} & \text{if } (x, y) \neq (0, 0), \\
1 & \text{if } (x, y) = (0, 0).
\end{cases}
\]

8. For each of the following, decide whether the given set is open, closed, neither open nor closed, or both open and closed.

(a) \((3, 10)\) in \(\mathbb{R}\)

(b) \([-2, 5]\) in \(\mathbb{R}\)

(c) \(\{(x, y) : x^2 + y^2 < 4\}\) in \(\mathbb{R}^2\)

(d) \(\{(x, y) : x^2 + y^2 > 4\}\) in \(\mathbb{R}^2\)
(e) $\{(x, y) : x^2 + y^2 \leq 4\}$ in $\mathbb{R}^2$
(f) $\{(x, y) : x^2 + y^2 = 4\}$ in $\mathbb{R}^2$
(g) $\{(x, y, z) : -1 < x < 1, -2 < y < 3, 2 < z < 5\}$ in $\mathbb{R}^3$
(h) $\{(x, y) : -3 < x \leq 4, -2 \leq y < 1\}$ in $\mathbb{R}^2$

9. Give an example of a subset of $\mathbb{R}$ which is neither open nor closed.

10. Is it possible for a subset of $\mathbb{R}^2$ to be both open and closed? Explain.