The Calculus of Functions of Several Variables

Section 1.3

The Cross Product

As we noted in Section 1.1, there is no general way to define multiplication for vectors in $\mathbb{R}^n$, with the product also being a vector of the same dimension, which is useful for our purposes in this book. However, in the special case of $\mathbb{R}^3$ there is a product which we will find useful. One motivation for this product is to consider the following problem: Given two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, not parallel to one another, find a third vector $\mathbf{w} = (w_1, w_2, w_3)$ which is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$. Thus we want $\mathbf{w} \cdot \mathbf{x} = 0$ and $\mathbf{w} \cdot \mathbf{y} = 0$, which means we need to solve the equations

$$\begin{aligned} x_1 w_1 + x_2 w_2 + x_3 w_3 &= 0 \\ y_1 w_1 + y_2 w_2 + y_3 w_3 &= 0 \end{aligned} \tag{1.3.1}$$

for w_1, w_2, and w_3. Multiplying the first equation by y_3 and the second by x_3 gives us

$$\begin{aligned} x_1 y_3 w_1 + x_2 y_3 w_2 + x_3 y_3 w_3 &= 0 \\ x_3 y_1 w_1 + x_3 y_2 w_2 + x_3 y_3 w_3 &= 0. \end{aligned} \tag{1.3.2}$$

Subtracting the second equation from the first, we have

$$(x_1 y_3 - x_3 y_1) w_1 + (x_2 y_3 - x_3 y_2) w_2 = 0. \tag{1.3.3}$$

One solution of (1.3.3) is given by setting

$$\begin{aligned} w_1 &= x_2 y_3 - x_3 y_2 \\ w_2 &= -(x_1 y_3 - x_3 y_1) = x_3 y_1 - x_1 y_3. \end{aligned} \tag{1.3.4}$$

Finally, from the first equation in (1.3.1), we now have

$$x_3 w_3 = -x_1 (x_2 y_3 - x_3 y_2) - x_2 (x_3 y_1 - x_1 y_3) = x_1 x_3 y_2 - x_2 x_3 y_1, \tag{1.3.5}$$

from which we obtain the solution

$$w_3 = x_1 y_2 - x_2 y_1. \tag{1.3.6}$$

The choices made in arriving at (1.3.4) and (1.3.6) are not unique, but they are the standard choices which define $\mathbf{w}$ as the *cross* or *vector product* of $\mathbf{x}$ and $\mathbf{y}$.

Definition Given vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, the vector

$$\mathbf{x} \times \mathbf{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1) \tag{1.3.7}$$

is called the *cross product*, or *vector product*, of $\mathbf{x}$ and $\mathbf{y}$.

Example If $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, -1, 1)$, then

$$\mathbf{x} \times \mathbf{y} = (2 + 3, 3 - 1, -1 - 2) = (5, 2, -3).$$

Note that

$$\mathbf{x} \cdot (\mathbf{x} \times \mathbf{y}) = 5 + 4 - 9 = 0$$

and

$$\mathbf{y} \cdot (\mathbf{x} \times \mathbf{y}) = 5 - 2 - 3 = 0,$$

showing that $\mathbf{x} \perp (\mathbf{x} \times \mathbf{y})$ and $\mathbf{y} \perp (\mathbf{x} \times \mathbf{y})$ as claimed. It is also interesting to note that

$$\mathbf{y} \times \mathbf{x} = (-3 - 2, 1 - 3, 2 + 1) = (-5, -2, 3) = -(\mathbf{x} \times \mathbf{y}).$$

This last calculation holds in general for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$.

Proposition Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are vectors in $\mathbb{R}^3$ and α is any real number. Then

$$\mathbf{x} \times \mathbf{y} = -(\mathbf{y} \times \mathbf{x}), \tag{1.3.8}$$

$$\mathbf{x} \times (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \times \mathbf{y}) + (\mathbf{x} \times \mathbf{z}), \tag{1.3.9}$$

$$(\mathbf{x} + \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \times \mathbf{z}) + (\mathbf{y} \times \mathbf{z}), \tag{1.3.10}$$

$$\alpha(\mathbf{x} \times \mathbf{y}) = (\alpha\mathbf{x}) \times \mathbf{y} = \mathbf{x} \times (\alpha\mathbf{y}), \tag{1.3.11}$$

and

$$\mathbf{x} \times \mathbf{0} = \mathbf{0}. \tag{1.3.12}$$

Verification of these properties is straightforward and will be left to Problem 10. Also, notice that

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \tag{1.3.13}$$

$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \tag{1.3.14}$$

and

$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2; \tag{1.3.15}$$

that is, the cross product of two standard basis vectors is either the other standard basis vector or its negative. Moreover, note that in these cases the cross product points in the direction your thumb would point if you were to wrap the fingers of your right hand from the first vector to the second. This is in fact always true and results in what is known as the *right-hand rule* for the orientation of the cross product, as shown in Figure 1.3.1. Hence given two vectors $\mathbf{x}$ and $\mathbf{y}$, we can always determine the direction of $\mathbf{x} \times \mathbf{y}$; to

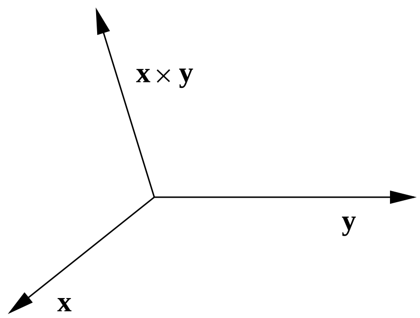

Figure 1.3.1 The right-hand rule

completely identify $\mathbf{x} \times \mathbf{y}$ geometrically, we need only to know its length. Now if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$, then

$$\begin{aligned}
\|\mathbf{x} \times \mathbf{y}\|^2 &= (x_2y_3 - x_3y_2)^2 + (x_3y_1 - x_1y_3)^2 + (x_1y_2 - x_2y_1)^2 \\
&= x_2^2y_3^2 - 2x_2x_3y_2y_3 + x_3^2y_2^2 + x_3^2y_1^2 - 2x_1x_3y_1y_3 + x_1^2y_3^2 + x_1^2y_2^2 \\
&\qquad - 2x_1x_2y_1y_2 + x_2^2y_1^2 \\
&= (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2) \\
&\qquad - (2x_2x_3y_2y_3 + 2x_1x_3y_1y_3 + 2x_1x_2y_1y_2) \\
&= (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2) - (x_1y_1 + x_2y_2 + x_3y_3)^2 \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2 \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2 - (\|\mathbf{x}\|\|\mathbf{y}\|\cos(\theta))^2 \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2(1 - \cos^2(\theta)) \\
&= \|\mathbf{x}\|^2\|\mathbf{y}\|^2\sin^2(\theta). \qquad (1.3.16)
\end{aligned}$$

Taking square roots, and noting that $\sin(\theta) \geq 0$ since, by the definition of the angle between two vectors, $0 \leq \theta \leq \pi$, we have the following result.

Proposition If θ is the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$, then

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\|\sin(\theta). \qquad (1.3.17)$$

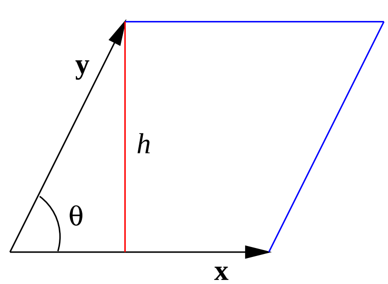

Figure 1.3.2 Height of the parallelogram is $h = \|\mathbf{y}\|\sin(\theta)$

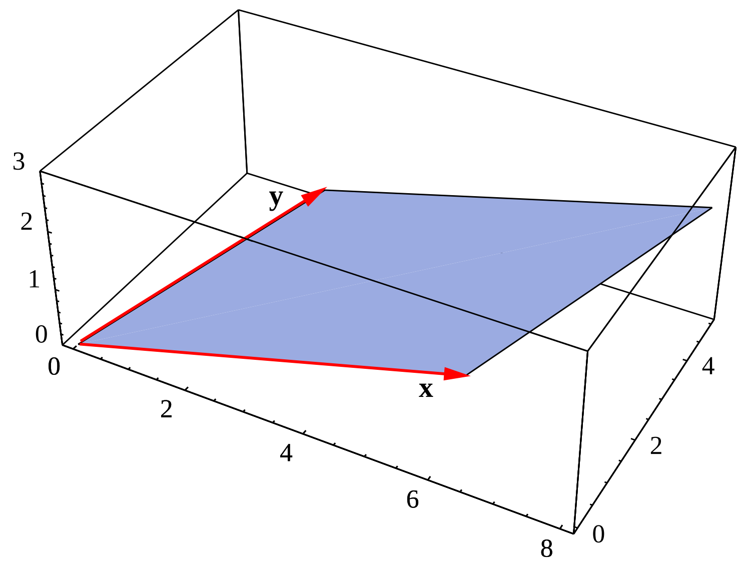

Figure 1.3.3 Parallelogram with vertices at $(0,0,0)$, $(6,1,1)$, $(8,5,2)$, and $(2,4,1)$

The last theorem has several interesting consequences. One of these comes from recognizing that if we draw a parallelogram with $\mathbf{x}$ and $\mathbf{y}$ as adjacent sides, as in Figure 1.3.2, then the height of the parallelogram is $\|\mathbf{y}\|\sin(\theta)$, where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Hence the area of the parallelogram is $\|\mathbf{x}\|\|\mathbf{y}\|\sin(\theta)$, which by (1.3.17) is $\|\mathbf{x}\times\mathbf{y}\|$.

Proposition Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors in $\mathbb{R}^3$. Then the area of the parallelogram which has $\mathbf{x}$ and $\mathbf{y}$ for adjacent sides is $\|\mathbf{x}\times\mathbf{y}\|$.

Example Consider the parallelogram P with vertices at $(0,0,0)$, $(6,1,1)$, $(8,5,2)$, and $(2,4,1)$. Two adjacent sides are specified by the vectors $\mathbf{x}=(6,1,1)$ and $\mathbf{y}=(2,4,1)$ (see Figure 1.3.3), so the area of P is

$$\|\mathbf{x}\times\mathbf{y}\| = \|(1-4, 2-6, 24-2)\| = \|(-3,-4,22)\| = \sqrt{509}.$$

See Figure 1.3.4 to see the relationship between $\mathbf{x}\times\mathbf{y}$ and P.

Example Consider the parallelogram P in the plane with vertices at $(1,1,)$, $(3,2)$, $(4,4)$, and $(2,3)$. Two adjacent sides are given by the vectors from $(1,1)$ to $(3,2)$, that is

$$\mathbf{x} = (3,2)-(1,1) = (2,1),$$

and from $(1,1)$ to $(2,3)$, that is,

$$\mathbf{y} = (2,3)-(1,1) = (1,2).$$

See Figure 1.3.5. However, since these vectors are in $\mathbb{R}^2$, not in R^3, we cannot compute their cross product. To get around this, we consider the vectors $\mathbf{w}=(2,1,0)$ and $\mathbf{v}=(1,2,0)$ which are adjacent sides of the same parallelogram viewed as lying in $\mathbb{R}^3$. Then the area of P is given by

$$\|\mathbf{w}\times\mathbf{v}\| = \|(0,0,4-1)\| = \|(0,0,3)\| = 3.$$

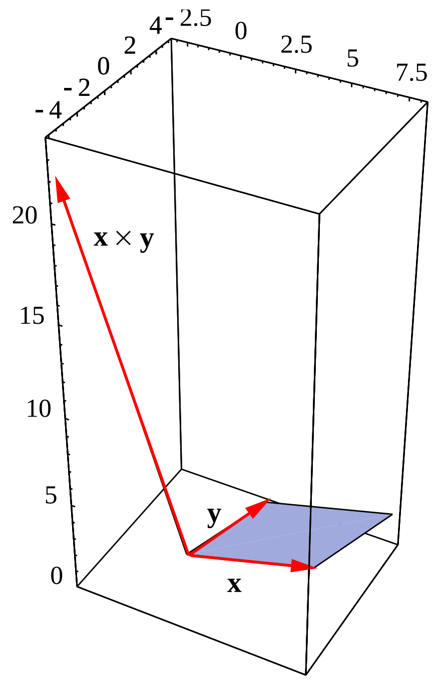

Figure 1.3.4 Parallelogram with adjacent sides $\mathbf{x} = (6, 1, 1)$ and $\mathbf{y} = (2, 4, 1)$

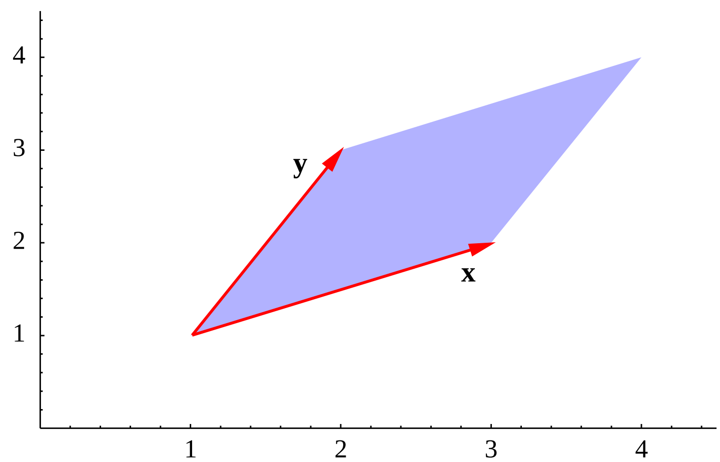

Figure 1.3.5 Parallelogram with vertices at $(1, 1)$, $(3, 2)$, $(4, 4)$, and $(2, 3)$

It is easy to extend the result of the previous theorem to computing the volume of a parallelepiped in $\mathbb{R}^3$. Suppose $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are adjacent edges of parallelepiped P, as shown in Figure 1.3.6. Then the volume V of P is $\|\mathbf{x} \times \mathbf{y}\|$, which is the area of the base, multiplied by the height of P, which is the length of the projection of $\mathbf{z}$ onto $\mathbf{x} \times \mathbf{y}$. Since the latter is equal to

$$\left| \mathbf{z} \cdot \frac{\mathbf{x} \times \mathbf{y}}{\|\mathbf{x} \times \mathbf{y}\|} \right|,$$

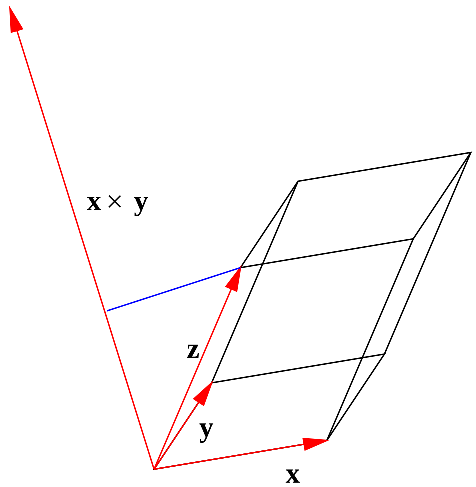

Figure 1.3.6 Parallelepiped with adjacent edges **x**, **y**, and **z**

we have

$$V = \|\mathbf{x} \times \mathbf{y}\| \left| \mathbf{z} \cdot \frac{\mathbf{x} \times \mathbf{y}}{\|\mathbf{x} \times \mathbf{y}\|} \right| = |\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})|. \tag{1.3.18}$$

Proposition The volume of a parallelepiped with adjacent edges $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ is $|\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})|$.

Definition Given three vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ in $\mathbb{R}^3$, the quantity $\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})$ is called the *scalar triple product* of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x}$.

Example Let $\mathbf{x} = (1, 4, 1)$, $\mathbf{y} = (-3, 1, 1)$, and $\mathbf{z} = (0, 1, 5)$ be adjacent edges of parallelepiped P (see Figure 1.3.7). Then

$$\mathbf{x} \times \mathbf{y} = (4 - 1, -3 - 1, 1 + 12) = (3, -4, 13),$$

so

$$\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) = 0 - 4 + 65 = 61.$$

Hence the volume of P is 61.

The final result of this section follows from (1.3.17) and the fact that the angle between parallel vectors is either 0 or π.

Proposition Vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ are parallel if and only if $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.

Note that, in particular, for any vector $\mathbf{x}$ in $\mathbb{R}^3$, $\mathbf{x} \times \mathbf{x} = \mathbf{0}$

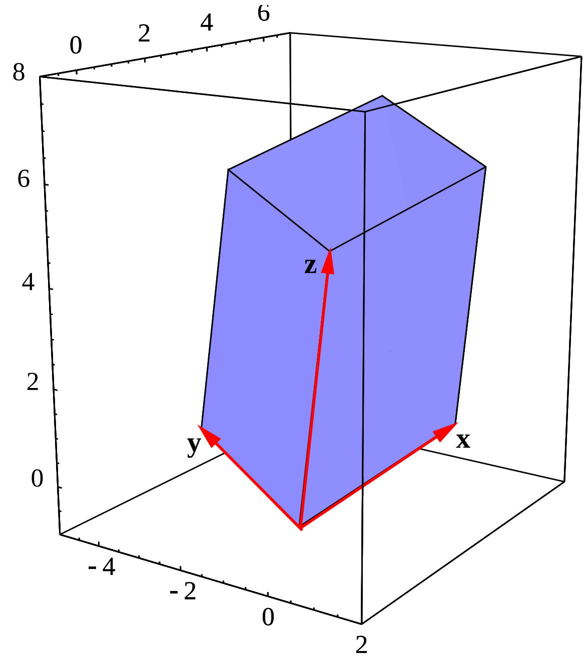

Figure 1.3.7 Parallelepiped with adjacent edges $\mathbf{x} = (1, 4, 1)$, $\mathbf{y} = (-3, 1, 1)$, $\mathbf{z} = (0, 1, 5)$

Problems

1. For each of the following pairs of vectors $\mathbf{x}$ and $\mathbf{y}$, find $\mathbf{x} \times \mathbf{y}$ and verify that $\mathbf{x} \perp (\mathbf{x} \times \mathbf{y})$ and $\mathbf{y} \perp (\mathbf{x} \times \mathbf{y})$.

   (a) $\mathbf{x} = (1, 2, -1)$, $\mathbf{y} = (-2, 3, -1)$
   (b) $\mathbf{x} = (-2, 1, 4)$, $\mathbf{y} = (3, 1, 2)$
   (c) $\mathbf{x} = (1, 3, -2)$, $\mathbf{y} = (3, 9, 6)$
   (d) $\mathbf{x} = (-1, 4, 1)$, $\mathbf{y} = (3, 2, -1)$

2. Find the area of the parallelogram in $\mathbb{R}^3$ that has the vectors $\mathbf{x} = (2, 3, 1)$ and $\mathbf{y} = (-3, 3, 1)$ for adjacent sides.

3. Find the area of the parallelogram in $\mathbb{R}^2$ that has the vectors $\mathbf{x} = (3, 1)$ and $\mathbf{y} = (1, 4)$ for adjacent sides.

4. Find the area of the parallelogram in $\mathbb{R}^3$ that has vertices at $(1, 1, 1)$, $(2, 3, 2)$, $(-2, 4, 4)$, and $(-3, 2, 3)$.

5. Find the area of the parallelogram in $\mathbb{R}^2$ that has vertices at $(2, -1)$, $(4, -2)$, $(3, 0)$, and $(1, 1)$.

6. Find the area of the triangle in $\mathbb{R}^3$ that has vertices at $(1, 1, 0$, $(2, 3, 1)$, and $(-1, 3, 2)$.

7. Find the area of the triangle in $\mathbb{R}^2$ that has vertices at $(-1, 2)$, $(2, -1)$, and $(1, 3)$.

8. Find the volume of the parallelepiped that has the vectors $\mathbf{x} = (1, 2, 1)$, $\mathbf{y} = (-1, 1, 1)$, and $\mathbf{z} = (-1, -1, 6)$ for adjacent sides.

9. A parallelepiped has base vertices at $(1, 1, 1)$, $(2, 3, 2)$, $(-2, 4, 4)$, and $(-3, 2, 3)$ and top vertices at $(2, 2, 6)$, $(3, 4, 7)$, $(-1, 5, 9)$, and $(-2, 3, 8)$. Find its volume.

10. Verify the properties of the cross product stated in (1.3.8) through (1.3.12).

11. Since $|\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y})|$, $|\mathbf{y} \cdot (\mathbf{z} \times \mathbf{x})|$, and $|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})|$are all equal to the volume of a parallelepiped with adjacent edges $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, they should all have the same value. Show that in fact
$$\mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x}) = \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}).$$
How do these compare with $\mathbf{z} \cdot (\mathbf{y} \times \mathbf{z})$, $\mathbf{y} \cdot (\mathbf{z} \times \mathbf{x})$, and $\mathbf{x} \cdot (\mathbf{z} \times \mathbf{y})$?

12. Suppose $\mathbf{x}$ and $\mathbf{y}$ are parallel vectors in $\mathbb{R}^3$. Show directly from the definition of the cross product that $\mathbf{x} \times \mathbf{y} = \mathbf{0}$.

13. Show by example that the cross product is not associative. That is, find vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ such that
$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) \neq (\mathbf{x} \times \mathbf{y}) \times \mathbf{z}.$$